# Prediction of the Flowfield in Laser Propulsion Devices

Charles L. Merkle*
*The Pennsylvania State University, University Park, Pennsylvania*

**The difficulties encountered in modeling laser heating in a propulsive nozzle are discussed. Two basic formulations, an approximate low Mach number analysis or the complete Navier-Stokes equations, can be used. The former reduces to a single ordinary differential equation for which the mass flow is an eigenvalue. This representation has been considered by several researchers, but the precision with which the eigenvalue must be determined makes it impractical to obtain solutions for some values of the parameters. The present paper contains the first reported solutions to the complete Navier-Stokes equations for this problem. These results, which were obtained with an implicit time-marching method, no longer contain an eigenvalue, but the complex, rapidly developing physics in the flowfield prevent convergence in some regimes. Solutions for laser absorption in hydrogen-seedant mixtures demonstrate that the position of the heat absorption region is controlled by the incoming beam intensity while the mass flux is determined by the nozzle throat area. These two methods are related to each other by showing that the eigenvalue analysis is a rational approximation to the Navier-Stokes equations.**

## Introduction

THE interaction between the incoming radiation and the flowing gas in a laser thruster is highly complex and strongly nonlinear. Improved understanding of this process is required if this potentially attractive propulsion concept is to be developed. In grasping this understanding, experimental studies must certainly be undertaken (Refs. 1 and 2 describe experiments currently underway) but analytical results are also necessary to guide and interpret the experimental studies, and to scale laboratory results to full-scale applications. The present paper addresses the current status of analyses of radiation-gasdynamic interactions in a laser thruster and presents some new results. Throughout the paper attention is focused on continuous wave (as opposed to pulsed) laser thrusters.

The general picture of a laser thruster is similar to that depicted in Fig. 1. Both the laser beam and the cool unheated gas enter the nozzle from the upstream end. As the gas flows toward the throat, it is heated by absorption of radiant energy causing it to accelerate. The converging nozzle walls also aid in accelerating the gas. As the gas nears the throat, it reaches sonic velocity and becomes supersonic downstream. (For space propulsion applications, it is assumed that the nozzle remains choked at all times.) Although the mixing and recombination phenomena that take place in the supersonic portion of the nozzle are of importance to propulsive performance, they do not affect the subsonic portion of the absorption/acceleration process. Consequently, these supersonic phenomena are omitted in the present discussion.

In a practical laser thruster the absorption heating will probably be restricted to the region near the centerline of the nozzle, leaving a cool, unheated layer of gas near the wall.[3] This pattern will produce a highly two-dimensional flowfield. The heat loss from the hot central core to the wall and the mixing between the core and the cool surrounding gas are phenomena of prime importance. Unfortunately, theoretical models are not yet sufficiently advanced to handle all these complications. Consequently, much of the emphasis in the present paper is directed toward one-dimensional formulations which assume uniform heating over the cross section. Similarly, the details of the flow entry are ignored so that emphasis can be placed on understanding the laser-gasdynamic interaction.

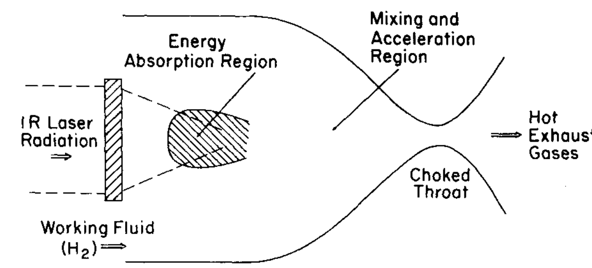


Fig. 1 Laser thruster.

The analytical models that are useful for describing this interaction include approximate formulations in which the mass flow emerges as an eigenvalue, as well as formulations which are expressed in terms of the complete unsteady Navier-Stokes equations. The approximate eigenvalue formulation has been applied only to interactions in constant area ducts and cannot be extended to laser heating in nozzles. Nevertheless, even for this simplified analysis, the present problem is sufficiently complex that the most straightforward, iterative technique fails to converge for all cases of interest.

The unsteady Navier-Stokes formulation is applicable to the more general problem of laser-gasdynamic interaction in a propulsive nozzle, but here again, current algorithms are adequate for only some laser propulsion conditions. Additional work on both solution methods is required to be able to model the entire parameter range of interest. In the present paper the characteristics of these two mathematical models are reviewed and a series of flowfield predictions, the first ever obtained with the full Navier-Stokes equations, are presented. Finally, the relationship between the two methods is described.

## Synopsis of Previous Work

The first analysis of the interaction between a laser beam and a flowing gas was reported by Raizer.[4] He considered a one-dimensional, doubly infinite problem. Using an approximation familiar to combustion analysis,[5] he replaced the momentum equation by the statement that pressure is a constant, thus reducing the equations of motion to a single ordinary differential equation. The solutions to this equation show that the mass flux is an eigenvalue of the problem, and that a laser supported plasma propagates at a unique speed for each incoming intensity. Raizer's analysis was extended by Kemp and co-workers[6,7] to include the effects of realistic

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*Professor, Mechanical Engineering Department. Member AIAA.

property variations. The incorporation of variable properties can change the peak temperatures in the plasma by as much as a factor of three. Thus, it is clear that physically realistic properties must be used if the predictions are to be quantitatively meaningful. For their calculations, Kemp et al. used the thermophysical properties of pure hydrogen and considered radiation at 10.6 $\mu$m. Additional analyses of the eigenvalue problem are given in Refs. 8 and 9.

Raizer's analysis has also been extended to two dimensions by Keefer and co-workers.[10,11] This extension requires some simplification of the gas dynamics but has produced the only two-dimensional temperature profile predictions thus far. The only attempts to include the effects of a nozzle downstream of the laser-gasdynamic interaction have been Refs. 12 and 13. In Ref. 12 the inverse problem was considered. A nozzle throat area whose choked-flow requirements precisely matched the mass flux from an eigenvalue solution was determined. In Ref. 13, a nozzle geometry was specified and a matching procedure was used to couple the eigenvalue solutions to corresponding one-dimensional nozzle flow solutions.

These approximate analyses are reviewed briefly below along with the exact Navier-Stokes formulation and their impact on the understanding of the laser propulsion problem is assessed.

## Absorption in a Constant Area Duct—the Eigenvalue Problem

Following Raizer's analysis,[4,5] we begin by considering the radiation-gasdynamic interaction in a constant area duct. By assuming that the Mach number is everywhere small, the momentum equation simplifies to the statement that pressure is constant and the equations of motion reduce to

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\lambda\frac{\mathrm{d}\theta}{\mathrm{d}X}\right)-\Lambda\frac{\mathrm{d}H}{\mathrm{d}X}=-QK\exp\left\{-\int_0^x K\,\mathrm{d}X\right\} \tag{1}$$

The nondimensional temperature $\theta$ and enthalpy $H$ are defined as

$$\theta=\frac{T-T_{-\infty}}{T_0-T_{-\infty}} \quad \text{and} \quad H=\frac{h}{c_{p_0}(T_0-T_{-\infty})} \tag{2}$$

where $T_0$ and $c_{p_0}$ are reference values of the temperature and specific heat, and $T_{-\infty}$ is the temperature far upstream of the heating zone. The longitudinal coordinate $X$ and absorptivity $K$ are nondimensionalized by a reference absorptivity $k_0$. The thermal conductivity $\lambda$ is nondimensionalized by $\lambda_0$. All reference properties are evaluated at temperature $T_0$. It is noted that for the temperature range of interest, the enthalpy $h$ is a function of both temperature and pressure.

The remaining two quantities in Eq. (1) represent the two parameters in the problem. These are the nondimensional mass flux, $\Lambda$, and the nondimensional heat addition, $Q$. These are defined as

$$\lambda=\frac{\dot{m}c_{p_0}}{\lambda_0 k_0} \qquad Q=\frac{I_0}{\lambda_0 k_0(T_0-T_{-\infty})} \tag{3}$$

where $I_0$ is the energy density of the incoming beam, and $\dot{m}$ is the mass flux per unit area.

Because of the variable coefficients, Eq. (1) must be solved numerically; however, more understanding can be gained by considering its solution in the constant coefficient limit. If the thermal conductivity and absorptivity are constants, Eq. (1) reduces to

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}X^2}-\Lambda\frac{\mathrm{d}\theta}{\mathrm{d}X}=0 \qquad \text{for } X\le 0 \tag{4a}$$

and

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}X^2}-\Lambda\frac{\mathrm{d}\theta}{\mathrm{d}X}=-KQe^{-KX} \qquad \text{for } X\ge 0 \tag{4b}$$

where we have taken $K=0$ for $\theta<\theta_i$ and $K=\text{const}$ for $\theta>\theta_i$, and where $X=0$ is set at the location where $\theta=\theta_i$.

The solution to Eq. (4) is

$$\theta=\theta_i e^{\Lambda X} \qquad X\le 0 \tag{5a}$$

and

$$\theta=\frac{Q}{\Lambda}-\frac{Q}{K+\Lambda}e^{-KX}+\left(\theta_i-\frac{KQ/\Lambda}{K+\Lambda}\right)e^{\Lambda X} \qquad X\ge 0 \tag{5b}$$

A boundedness condition dictates that the coefficient of the positive exponential in Eq. (5b) must vanish. This condition leads to a unique mass flux requirement:

$$\Lambda=\frac{K}{2}\left[\left(1+\frac{4Q}{K\theta_i}\right)^{\frac{1}{2}}-1\right] \tag{6a}$$

For most laser propulsion problems, $Q\gg 1$ so the mass flux becomes approximately

$$\Lambda\simeq(KQ/\theta_i)^{\frac{1}{2}} \tag{6b}$$

One method which has been used for solving the variable property equation[6,7,13] has been to guess the mass flux $\Lambda$ and integrate numerically from $X=0$ until a downstream condition is reached which either verifies the initial guess or indicates the need for a new one. The difficulties with this shooting technique arise from the term analogous to the increasing exponential in Eq. (5b). To demonstrate this difficulty, we have applied this technique to the constant coefficient representation, Eq. (4). The exact solution to Eq. (4) [Eq. (5) with mass flux given by Eq. (6a)] was first tabulated and is given in Fig. 2. Then attempts were made to reproduce this exact solution by solving Eq. (4) numerically starting from $X=0$, using an assumed value of the mass flux which differed from the correct mass flux by a (small) prescribed amount. The results of these numerical solutions are also included on Fig. 2. For each value of $\Lambda$, the numerical integration proceeds closely along the exact solution until some point where it diverges rapidly in an exponential sense from the correct solution. This divergence is in the positive or negative direction depending upon whether the assumed value for $\Lambda$ is smaller or larger than the correct value. The location at which this divergence occurs depends upon the magnitude of the error in $\Lambda$, and upon the value of $Q$. For $Q=100$, choosing $\Lambda$ accurate to 0.1% yields a numerical integration which is accurate to about $X=0.5$, a location where the temperature has risen to nearly its final value. Decreasing the error in $\Lambda$ to one part in $10^6$ causes the solution to behave properly to $X=1.25$. An error of one part in $10^8$ yields a proper solution

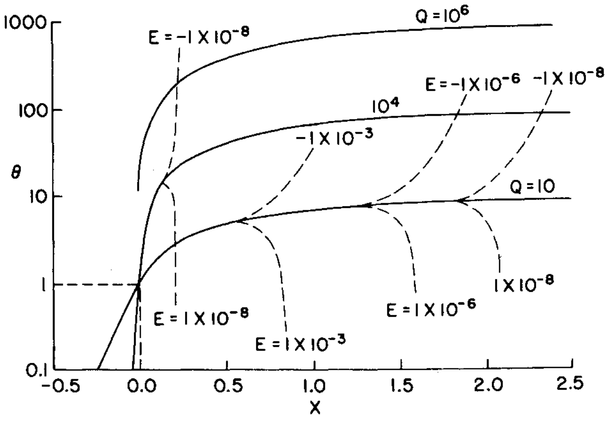


**Fig. 2 Exact and numerical solution of eigenvalue problem showing effect of error in mass flow on numerical results.**

to about $X = 1.75$. Comparison with the exact solution shows that any one of these guesses would give an acceptable solution to the problem if the integration were terminated just before the solution begins to diverge.

A completely different conclusion is obtained for the case where $Q = 10^4$. For this case, an error of one part in $10^8$ causes the exponential divergence from the correct solution to begin around $X = 0.15$ where the temperature is rising rapidly. Thus, at this high heat addition rate, even an accuracy of one part in $10^8$ is insufficient to integrate to a location where most of the energy has been absorbed. This divergence is independent of the step size and cannot be avoided by using smaller steps.

The reason for this exponential divergence is clearly the increasing exponential $e^{\Lambda X}$ in Eq. (5b). This exponential will appear in the numerical solution unless $\Lambda$ is precisely correct. As long as this coefficient is not identically zero, this increasing exponential will eventually give rise to the divergence noted above.

Although the variable coefficient problem is quantitatively different from these examples, its solutions behave qualitatively the same. In particular, as the value of $Q$ gets larger (which implies that the laser intensity is large, or the thermal conductivity or absorptivity is small) the precision required in the guess of the eigenvalue, $\Lambda$, becomes intolerably restrictive. Only problems in certain regimes can be solved numerically by this shooting technique. Numerical integrations of the variable property equation in Ref. 12 indicated that the shooting method worked reliably at a wavelength of 10.6 $\mu$m. Later attempts to extend these calculations to short wavelength radiation at 1.06 $\mu$m have thus far proved fruitless for the reasons noted above. Alternate techniques for solving the eigenvalue problem under these conditions are needed.

An alternative method for solving the eigenvalue equation can be obtained by approximating the variable properties in a step-wise fashion such that the properties remain constant over a number of small temperature ranges, but jump discontinuously between. This results in a nonlinear system of coupled equations for the eigenvalue, but appears to bypass the necessity for determining the eigenvalue to very rigid tolerances.

## The Eigenvalue Problem in Two Dimensions

The two-dimensional formulation of the eigenvalue problem again approximates the pressure as a constant (in both directions). The fluid dynamics are further simplified by assuming the flow is one-dimensional and the cross-stream velocity component $v$ is identically zero. No one has yet attempted solutions of the two-dimensional problem with complete property variations. Accordingly, we consider the constant coefficient case, for which the equations of motion again reduce to a single equation

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} - \Lambda\frac{\partial\theta}{\partial X} = -QKIe^{-KX} \tag{7}$$

(For purposes of understanding the characteristics of the solution, we have considered the planar case. Keefer[10,11] considered the more realistic axisymmetric geometry along with slightly more relaxed property approximations. The restriction to planar geometries replaces the Bessel series with the more familiar Fourier series, but does not affect the mathematical character of the solution.)

Equation (7) can be solved in closed form by the method of separation of variables. For the simplest case of an incoming beam whose intensity variation over the cross section is a cosine function,

$$I(0,y) = I_0\cos(\pi Y/2W) \tag{8}$$

where $W$ is the half-width of the beam, we have

$$\theta = \theta_{00}\cos AY\exp\{(\Lambda + B)X/2\} \qquad X \le 0 \tag{9a}$$

and

$$\theta = \cos AY\left\{\frac{QK}{K^2 + \Lambda K - A^2}\left[\frac{2K+\Lambda+B}{2B}\exp\left[\frac{(\Lambda - B)X}{2}\right] - \exp(-KX)\right] + \left[\theta_{00} - \frac{QK}{K^2+\Lambda K - A^2}\,\frac{2K+\Lambda-B}{2B}\right]\exp\left[\frac{(\Lambda+B)X}{2}\right]\right\} \tag{9b}$$

In these equations, $\theta_{00}$ is the temperature at $X = Y = 0$, $A = \pi/(2W)$, and $B = (\Lambda^2 + 4A^2)^{\frac{1}{2}}$. (Note that in this two-dimensional solution the absorptivity of the gas must jump from zero to $K$ at $X = 0$, whereas in the one-dimensional problem the jump can be taken as a function of temperature instead of position.) The solution to the two-dimensional problem, Eq. (9b), again contains an eigenvalue for the mass flow which arises because the coefficient of the second exponential in Eq. (9b) must go to zero. We also note that in the limit as the width $W$ of the laser beam becomes infinite, Eqs. (9a) and (9b) reduce to Eqs. (6a) and (6b) as expected. For solutions to the two-dimensional problem the reader is referred to Refs. 10 and 11.

It is noted that the complete variable coefficient problem can again be solved numerically (without restricting to separable solutions) by guessing a value for the mass flux $\Lambda$ and integrating in $X$ until divergence is reached, then improving the guess on $\Lambda$ and repeating this process. Nevertheless, it is noted that each iteration now requires the numerical solution of a partial differential equation (pde), and even though the pde is an initial value problem, it would appear that numerical solutions of the variable coefficient version of Eq. (7) are more involved than the approximate results would justify.

## Formulation as an Unsteady Problem—One Dimension

A more exact formulation of the radiation-gasdynamic interaction in a laser propulsion system is obtained by using the complete Navier-Stokes equations. To facilitate the solution, we consider the unsteady formulation from which we can obtain the steady solution as the limit of a time-marching calculation. To incorporate the effect of variable area, we use the quasi-one-dimensional formulation:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S \tag{10}$$

where

$$U = A\begin{pmatrix}\rho\\ \rho u\\ e\end{pmatrix} \quad F = \begin{pmatrix}\rho u\\ \rho u^2 + \sigma\\ (e+\sigma)u - kT_X\end{pmatrix} \quad S = \begin{pmatrix}0\\ p\,\mathrm{d}A/\mathrm{d}X\\ -KIA\end{pmatrix} \tag{11}$$

where $\sigma = p - (\lambda + 2\mu)\,\mathrm{d}u/\mathrm{d}x$; $A$ is the local flow area; and $e$ is the total energy, $= \rho\epsilon + \rho u^2/2$ ($\epsilon$ is the internal energy). The system is closed by an appropriate equation of state. This equation can be solved by discretizing in time and space and using an implicit time dependent technique such as those of MacCormack[14] or Beam and Warming.[15] Note that although radiative transfer has been ignored in the present formulation, it may indeed represent an important transport mechanism near the front of the absorbing region,[9] and should eventually be added to this formulation.

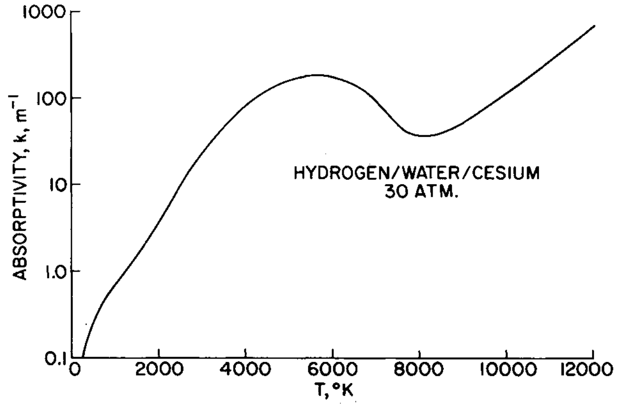


Fig. 3 Absorptivity of hydrogen, cesium, and water vapor mixture (0.98, 0.01, 0.01) at 30 atm (from calculations in Ref. 7).

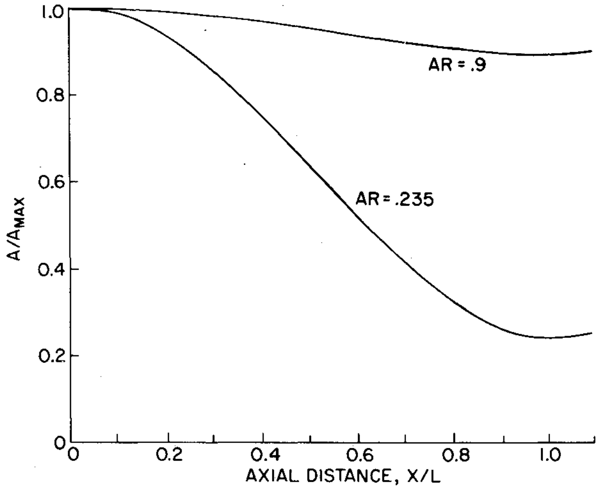


Fig. 4 Nozzle geometries for two nozzles used in time-marching calculations.

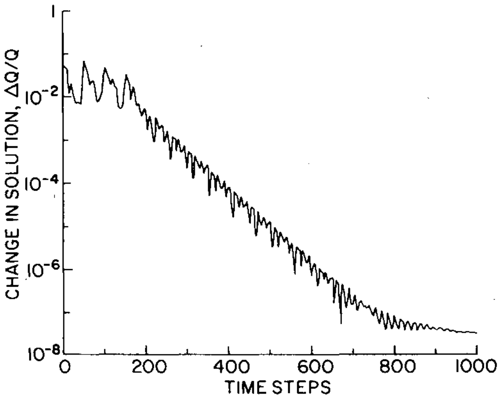


Fig. 5 Solution convergence vs time step for an area ratio of 0.235 and initial intensity $I_0 = 1 \times 10^6$ W/cm$^2$.

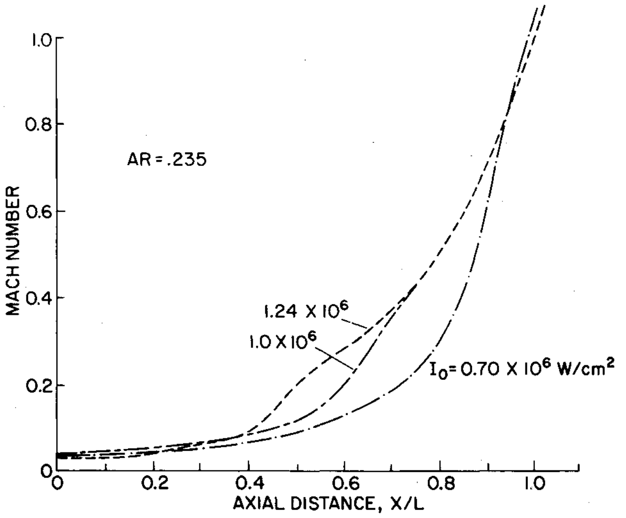


Fig. 6 Axial variation of Mach number in a nozzle of area ratio 0.235 for three values of incoming laser intensity.

This computational procedure, which is very different from the eigenvalue approach described above, is completely analogous to an experimental realization. The calculation starts by specifying the stagnation pressure and temperature at the entry plane, along with the incoming laser intensity. The downstream conditions are specified by requiring supersonic flow downstream of the throat. The calculation then proceeds forward in time holding these boundary conditions fixed. The mass flux through the nozzle automatically seeks its choked value which is properly determined by conditions at the throat. The eigenvalue for the mass flux through the plasma no longer exists (see the next section).

Although in principle this computation is straightforward and can be performed routinely for numerous problems, there are a number of characteristics of the present problem which can prevent convergence in many cases.[16] Numerical difficulties include: 1) the extremely steep temperature gradients associated with the laser propulsion problem; 2) the dramatic variations in the properties; 3) the strong source terms; 4) the stiff eigenvalues encountered when the entering Mach number is driven to very low values by large heating rates and small throat areas; and 5) the strong nonlinear coupling between the radiation and the gas dynamics. Numerous steps can be taken to overcome these problems, but current algorithms are not sufficient to obtain solutions for all parameter ranges of interest.

Some representative solutions of the full Navier-Stokes equations have been obtained for laser absorption in a hydrogen-seedant mixture flowing through a choked, convergent-divergent nozzle. The gas mixture contained 1% cesium and 1% water vapor. The absorptivity of this mixture has been determined by Kemp and Lewis[7] and is shown in Fig. 3. The thermal conductivity and enthalpy for these calculations were taken as those of equilibrium hydrogen at 30 atm. The nozzle geometries for which the results have been computed are given in Fig. 4. In keeping with the quasi-one-dimensional formulation, the laser beam was assumed to converge at a rate equal to the nozzle convergence. Thus, the unattenuated laser intensity was a maximum at the throat. This convergence does not accurately represent the true focusing of a beam, but nevertheless this represents a first attempt to include the all-important beam focusing in a one-dimensional problem.

The solutions were obtained by specifying the stagnation pressure, $p^0 = 30$ atm, and stagnation temperature, $T^0 = 500$ K, at the entrance to the nozzle and an approximate initial condition. The calculations then proceeded by marching in time until the change per time step became so small that the solution was no longer affected. A record of this convergence is shown in Fig. 5 for a typical case. As can be seen, the solution converges to very tight tolerances after about 200 time steps.

The converged Mach number profiles for three different beam intensities in a nozzle with area ratio of 0.235 (see Fig. 4) are given in Fig. 6. These profiles are all quite similar. The Mach number accelerates smoothly through the heating zone and the throat and reaches supersonic values downstream. For these three cases, all energy in the beam was absorbed. Mass fluxes of nominally 100 kg/m$^2$/s were predicted with very little variation with $I_0$, as will be noted later.

The variation of stagnation enthalpy for these three laser intensities is shown in Fig. 7. This figure more clearly shows

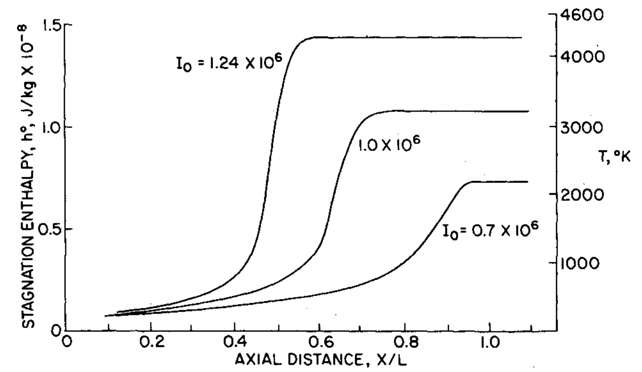


**Fig. 7 Axial variation of stagnation enthalpy and temperature for three values of incoming laser intensity. Nozzle area ratio 0.235.**

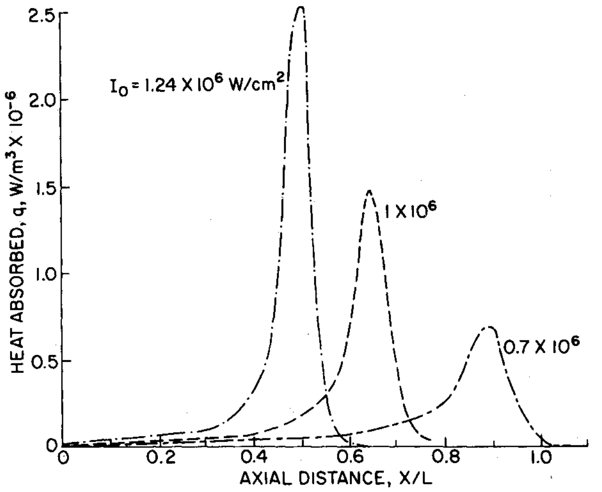


**Fig. 8 Axial variation of heat absorbed per unit volume for three values of incoming laser intensity. Nozzle area ratio 0.235.**

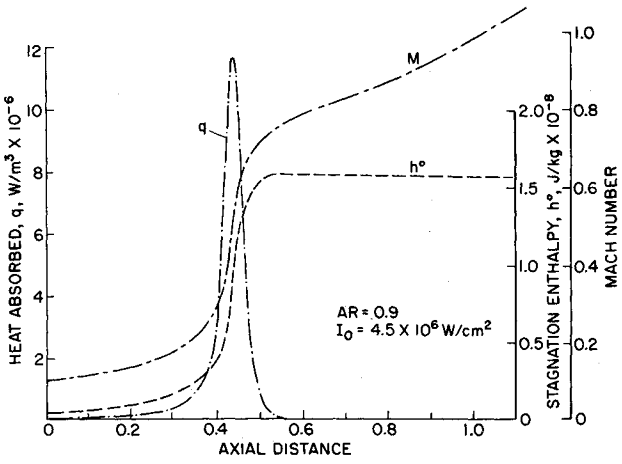


**Fig. 9 Axial variation of Mach number, stagnation enthalpy, and heat absorbed. Nozzle area ratio 0.9; incoming laser intensity, $4.5 \times 10^6$ W/cm$^2$.**

the effect of the laser intensity $I_0$ on the flowfield. As $I_0$ is increased, the plasma region moves further upstream of the throat and the peak temperature gets higher. This effect is also seen for Fig. 8, which shows the local heat absorption rate (per unit volume) $q$ for each of the three cases. Again at the higher intensities the heat is added further upstream of the throat and the rate of heat addition gets larger.

Similar calculations for a nozzle with an area ratio of 0.9 are shown in Fig. 9. With the larger throat area the rate of heat

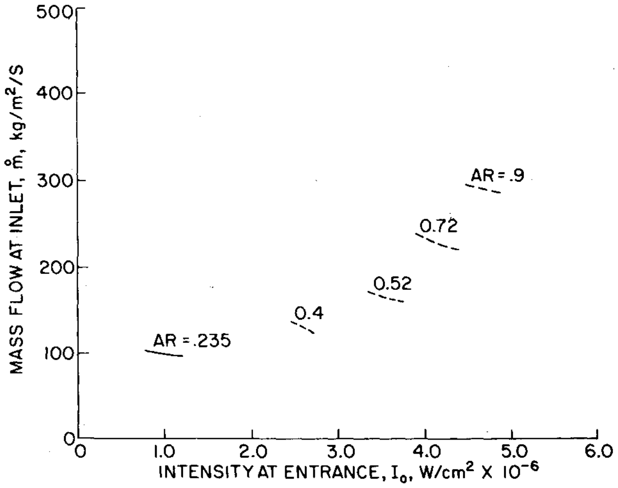


**Fig. 10 Mass flux per unit area at inlet vs incoming laser intensities for nozzles with various throat area sizes.**

addition is much higher, but the peak temperatures remain nearly the same as for the 0.235 area ratio case.

Finally, the mass flux through the nozzle for the various throat areas and incoming laser intensities is given in Fig. 10. (Since the inlet area is the same for all cases, this laser intensity is essentially the incoming laser power, but it must be expressed as a power per unit area because of the ill-defined nature of the cross-sectional area in a one-dimensional approximation.) Figure 10 clearly shows it is the throat area, not the laser power, which controls the mass flow. The mass flow is a weak function of the laser power while it is a strong function of the size of the throat area.

## Formulation of the Two-Dimensional Problem

The two-dimensional formulation of the problem is

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = S \tag{12}$$

where $U$, $E$, and $F$ take on their standard form,[17] and where the source term $S$ is somewhat simplified from the one-dimensional form:

$$S = (0, 0, 0, -KIA)^T \tag{13}$$

The methods referenced above can be directly extended to this two-dimensional formulation.

By extending to two dimensions, the unsteady formulation allows the unattenuated beam to be focused by appropriate lenses, it allows realistic cross-sectional variations in the beam intensity, and it allows mixing between the heated and unheated gases as well as realistic estimates of heat losses to the walls. Consequently, such two-dimensional calculations would be extremely useful in improving our understanding of these complex flow phenomena. As in the one-dimensional formulation, these calculations cannot be considered routine, and while existing algorithms are not adequate for all problems of interest, they are adequate for some. Accordingly, such applications are encouraged.

## Relation Between Eigenvalue Analysis and Navier-Stokes Equations

The Navier-Stokes equations contain the eigenvalue analysis as a subset. To show this, and to illustrate the physical character of the eigenvalue analysis, we expand the one-dimensional inviscid equations in terms of the Mach number. Since we are interested in low Mach number applications, we drop terms which are quadratic and higher in Mach number and retain only the linear expressions. The low Mach number approximation yields

$$\frac{\partial U'}{\partial t}+\frac{\partial E'}{\partial x}=S' \tag{14}$$

where the modified vectors are defined as

$$U'=\begin{pmatrix}\rho A\\ \rho uA\\ pA\end{pmatrix}\qquad E'=\begin{pmatrix}\rho uA\\ pA\\ \gamma puA\end{pmatrix}\qquad S'=\begin{pmatrix}0\\ p\,\mathrm{d}A/\mathrm{d}x\\ -(\gamma-1)KIA\end{pmatrix} \tag{15}$$

To ascertain the mathematical character of these equations, we look for the characteristic directions. These are most easily obtained by computing the Jacobian of the flux vector, $E'$,

$$\frac{\partial E'}{\partial U'}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -c^2u & c^2 & \gamma u\end{pmatrix} \tag{16}$$

The eigenvalues of this matrix are obtained from the secular equation:

$$\lambda^3-\gamma u\lambda^2-c^2\lambda+uc^2=0 \tag{17}$$

Although this equation is not easily factored, it is possible to obtain approximate roots by again using the low Mach number approximation. So doing yields the approximate roots

$$\lambda\simeq u;\quad \frac{\gamma-1}{2}\frac{u}{1-\gamma M}+c;\quad \frac{\gamma-1}{2}\frac{u}{1-\gamma M}-c \tag{18}$$

where $M$ is the Mach number.

The low Mach number approximation has changed the characteristic directions, but they remain qualitatively the same as in the full equations. The characteristics are all real (indicating the problem is still hyperbolic) and there are two downstream-running and one upstream-running characteristics.

For the constant area case we see that the pressure during a transient varies with both $X$ and time. However, as steady conditions are approached the unsteady formulation in Eq. (14) becomes identical to the approximate eigenvalue analysis, Eq. (1). Further, we see that the steady solution, along with the correct eigenvalue for the mass flux, is found naturally as the result of a transient. In the unsteady formulation the necessary downstream information is communicated to the solution by the upstream-running characteristic. This path of information transfer is lost in the steady analysis and consequently the mass flow must be determined as an eigenvalue to incorporate the required downstream information.

In view of the difficulty encountered in obtaining eigenvalue solutions especially in certain regions, the use of this approximate unsteady analysis might prove to be useful. In addition, more complicated physical models, such as those which include nonequilibrium kinetics effects, will fit automatically into the unsteady formulation and may prove to be advantageous for determining this important effect. Furthermore, this approach may also prove to be applicable to the two-dimensional eigenvalue problem with variable coefficients, Eq. (7).

## Summary

The difficulties encountered in modeling the radiation-gasdynamic interaction in a propulsive nozzle are discussed. Two basic approaches are available for modeling these phenomena; an approximate method for which the mass flux is an eigenvalue, and the complete Navier-Stokes equations. The eigenvalue analysis has been used by several researchers in the past, but it is shown here that the precision with which the eigenvalue must be specified depends upon the physical parameters of the problem being solved. In some cases this requirement is so stringent that practical solutions are precluded. Extension to two dimensions appears to be practical only for constant coefficient cases.

The present paper contains the first application of the full Navier-Stokes equations. These results are based upon an implicit time-marching procedure. Although they are for a one-dimensional problem, the approach can be directly extended to two dimensions. There are, however, difficulties encountered with the time-marching procedure. The extremely steep temperature gradients, the dramatic variations in physical properties, the stiff eigenvalues, the strong nonlinearities, and the source terms cause currently available algorithms to fail for some laser propulsion cases. Numerical results not included herein have shown that both the level and the gradient of the absorptivity have important effects on convergence.

Representative solutions of laser absorption in a hydrogen-seedant mixture show that for a given nozzle geometry the Mach number profile is almost independent of the laser intensity, but that the temperature profile and the position at which the heat is absorbed are strong functions of the intensity. As the laser intensity is increased, the position of the plasma moves steadily upstream from near the throat region toward the nozzle inlet. Simultaneously the peak heat addition per unit volume increases. The results also show that the mass flow is a strong function of the throat area and a weak function of the incoming laser power.

The eigenvalue analysis is related to the full Navier-Stokes analysis by showing that the former can be obtained as a result of an expansion in Mach number. The eigenvalue in the approximate formulation serves the same purpose as the upstream-running characteristic in the unsteady version of the low Mach number approximation.

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